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SELF-SIMILARITY IN THE VAPOUR BUBBLE DYNAMICS

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Abstract—The covariance of equations governing the homogeneous vapour bubble dynamics with respect to the transformation of independent variables is stated. It leads to the self-similar time dependence of bubble radius in three cases: (1) when liquid pressure is constant, (2) when it varies with time and (3) in an ultrasonic field. This is confirmed by numerical calculations.

NOMENCLATURE

R,	bubble radius;
$\dot{R}$,	time derivative of the radius;
r,	radial coordinate in liquid;
v,	dimensionless space coordinate;
t,	time;
T,	temperature of liquid;
T',	vapour temperature;
T_∞,	temperature of liquid at infinity;
k,	thermal conductivity;
D,	thermal diffusivity;
L,	latent heat;
c_s,	specific heat of saturated vapour;
ρ,	liquid density;
ρ',	vapour density;
f,	ultrasonic frequency;
n,	number of ultrasonic cycles;
P_1,	amplitude of ultrasonic field;
P_0,	static pressure of liquid;
m,	transformation parameters;
$\bar{R}$,	average bubble radius per cycle of ultrasound;
P,	pressure of liquid.

INTRODUCTION

THE DEVELOPMENT of the technique of resonance and ultrasonic bubble chambers has required the theoretical consideration of the dynamics of a vapour bubble in liquid. Its behaviour is determined by such nonlinear processes as heat and mass transfer at the moving vapour–liquid interface, self-frequency pulsations, etc. which leads to a necessity of considering the set of nonlinear equations [1]. Its analytical solutions can be obtained in the simplest cases only [2, 3]. The basic results in this range have been obtained by numerical methods. Along with advantages this approach has a drawback that does not allow one to reveal the general properties of solutions. In this paper the property of self-similarity is stated for functions describing the radial motion of homogeneous vapour bubbles which allows one to reduce numerical calculations.

COLLAPSE OF THE VAPOUR BUBBLE IN A LIQUID AT CONSTANT PRESSURE

As is known [2, 3], the growth of a vapour bubble in a superheated liquid at constant pressure is governed mainly by the process of heat transfer between the liquid and the bubble, i.e. by the heat diffusion equation

$$R^2 \frac{\partial T}{\partial t} + R\dot{R}v(1-v^3)\frac{\partial T}{\partial v} = Dv^4 \frac{\partial^2 T}{\partial v^2}$$
$$0 \leqslant v = \frac{R(t)}{r} \leqslant 1 \tag{1}$$

with the initial and boundary conditions

$$T(0, v) = T_0(v); \qquad R(0) = R_0 \tag{2}$$

$$k\frac{\partial T(t, 1)}{\partial v} = -\rho' L\dot{R}R \tag{3}$$

$$T(t, 1) = T'; \qquad T(t, 0) = T_\infty. \tag{4}$$

By basing on the covariance of equation (1) and conditions (2)–(4) with respect to the transformation of independent variables

$$t \to \tilde{t} = mt \tag{5}$$

$$r^2 \to \tilde{r}^2 = mr^2 \tag{6}$$

it has been shown [3] that the growth of a vapour bubble at constant pressure in a superheated liquid is as follows:

$$R^2(t) = R^2(0) + At \tag{7}$$

where $A > 0$. In this case the solution of equation (1) has the self-similar form $T(t, v) = T(v)$. It has been shown [4] that the collapse of the vapour bubble at constant pressure is determined by relations (1)–(4). However, in this case $A < 0$ and the self-similar temperature distribution contradicts last boundary condition (4) and a solution in a more general form is needed.

In a general case relations (1)–(4) are covariant with respect to (5), (6) if the transformed solutions $\tilde{R}^2(t)$ and $\tilde{T}(t, v)$ are expressed by the original ones as follows:

$$\tilde{R}^2(t) = mR^2(t/m) \tag{8}$$

$$\tilde{T}(t, v) = T(t/m, v). \tag{9}$$

Condition (8) means that the functions $R^2(t)$ form a single-parameter self-similar family of curves with the center of similarity at the beginning of the coordinate system. Figure 1 shows the dependences $R^2(t)$ describing the collapse of bubbles at various initial radii in liquid hydrogen obtained by numerical integration of (1)–(4). It is seen that the self-similarity does take place. It is sufficient that the initial temperature distribution $T(0, \nu)$ can be arbitrary but fixed.

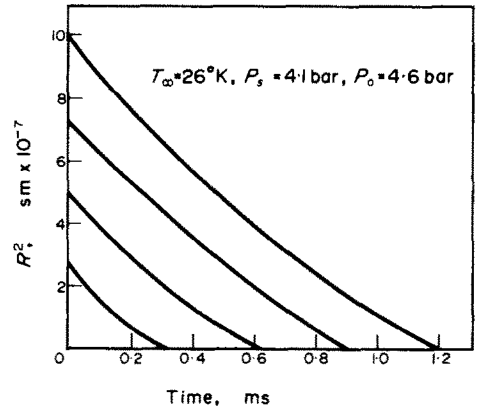

FIG. 1. Collapse of vapour bubbles of various radii in liquid hydrogen at constant pressure. Numerical solutions.

Consider equations (5) and (8) at the initial moment $t = 0$ and at the end of the bubble collapse t^*.

$$\tilde{R}^2(0) = mR^2(0), \qquad \tilde{t}^* = mt^*. \tag{10}$$

Excluding the transformation parameter, one obtains

$$\tilde{t}^* = \frac{t^*}{R^2(0)}\tilde{R}^2(0) \tag{11}$$

i.e. the bubble life-time is proportional to the initial radius squared, that is the result mentioned in reference [5].

BUBBLE BEHAVIOUR DURING THE CYCLE OF THE CLASSICAL BUBBLE CHAMBER

In bubble chambers the growth and collapse of vapour bubbles occurs at time-dependent pressure. Since the pressure changes slowly the inertial effects of the liquid can be neglected. Boundary condition (3) must be replaced by the following one:

$$\left(L\frac{\mathrm{d}\rho'}{\mathrm{d}T'} + c_s\rho'\right)\frac{\mathrm{d}T'}{\mathrm{d}t} = -\frac{3}{R^2}\left(k\frac{\partial T(t,1)}{\partial \nu} + \rho' LR\dot{R}\right). \tag{12}$$

The expression in the l.h.s. of equation (12) depends upon temperature only. The temperature itself is the function of liquid pressure $T' = F(P)$, where F denotes the vapour–liquid equilibrium curve. One can see that self-similarity $R^2(t)$ takes place if relations (8) and (9) are added by the proper pressure transformation

$$\tilde{P}(t) = P(t/m). \tag{13}$$

The solid curve in Fig. 2 shows the numerical solutions obtained with the values of thermodynamical

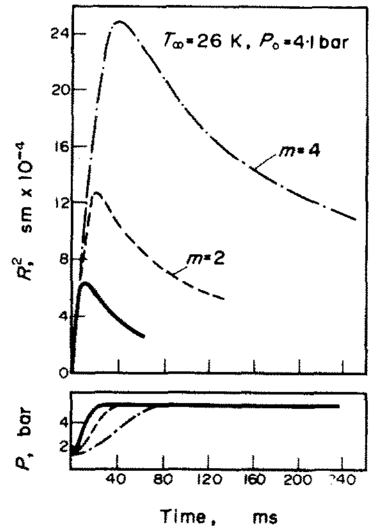

FIG. 2. Dependences $R^2(t)$ during the cycle of the liquid hydrogen bubble chamber at various regimes. Solid curve corresponds to experimental data [6]. Pressure dependences are shown at the lower plot.

parameters which correspond to the working conditions of the hydrogen bubble chamber [6]. Pressure variations in the chamber are shown in the lower plot. The dashed curves are numerical solutions corresponding to the transformed functions with $m = 2$ and $m = 4$.

It is seen that self-similarity of the functions $R^2(t)$ takes place indeed and it is of interest to obtain the experimental data corresponding to the transformed pressure variation $\tilde{P}(t)$.

SELF-SIMILARITY OF BUBBLE DYNAMICS IN THE ULTRASONIC BUBBLE CHAMBER

The set of equations underlying bubble dynamics in the ultrasonic field [1] includes along with (1), (12) the Rayleigh equation which takes into account the liquid inertia essential in rapid pressure variations. The Rayleigh equation is noncovariant with respect to transformations (5), (8).

However, it has been shown that the growth of a homogeneous vapour bubble in an underheated liquid in the ultrasonic field is determined by the rectified heat diffusion [1]. This allows the suggestions that the violation of equation covariance and consequently the self-similarity of $\tilde{R}^2(t)$ are not of importance at low frequencies. If liquid inertia is neglected, the vapour of pressure equals that of liquid which is determined as $P = P(f \times t)$.

Transformations (13) for such a function are as follows:

$$\tilde{P}(t) = P\left(\frac{f}{m}t\right) = P(\tilde{f}t) \tag{14}$$

$$\tilde{f} = \frac{f}{m}. \tag{15}$$

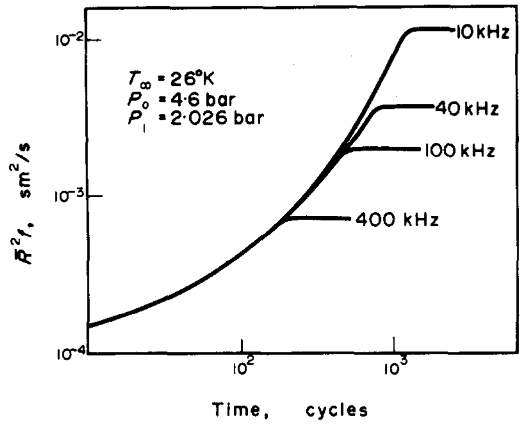

FIG. 3. Vapour bubble behaviour in liquid hydrogen for various ultrasonic frequencies. The differences in the shape of various curves shows the degree of the violation of self-similarity.

It means that the relationship between the functions $\bar{R}^2(t)$ corresponding to the frequencies f and $\tilde{f}$ take place which may be easily obtained if one excludes the parameter m from expressions (8) and (15). Passing over to time scaling in the cycles of ultrasound, one obtains the universal dependence

$$F(n) = \tilde{f}\tilde{\bar{R}}^2(n) = f\bar{R}^2(n). \qquad (16)$$

Figure 3 shows the computed solutions for the general system of equations [1] taking into account the factors violating the covariance for various frequencies. The initial radii of bubbles have been chosen to satisfy relation (16). The difference of the shape of separate curves from the universality can be considered as a measure of self-similarity violation of the functions $\bar{R}^2(t)$.

As it should be expected with the bubble growth, the inertia effects become of importance and the self-similarity is violated, the earlier the higher the frequencies of ultrasound. For sufficiently low frequencies the self-similar behaviour takes place and only one computer solution is needed to obtain the function $\bar{R}^2(t)$ for any frequency.

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